SOAL – SOAL DAN JAWABAN

PERMASALAHAN SISTEM DINAMIK

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Problem 2-4

Suppose that a very long conductor has been fixed in a vertical straight line : a constant current *I* passes though the conductor. A small conductor, length *l* and mass *m*, had been placed in vertical straight line; it has been fixed to a spring which can move horizontally. A constant current *I* passes through the small conductor. The small conductor will be put into motion in *x*-direction but it remains parallel to the long conductor; without deformation of the spring, its position is $x=0$. The fixed position of the long conductor is given by $x=a$. the equation of motion of the small conductor is

$$m\ddot{x}+kx-\frac{2Iil}{a-x}=0$$

With *k* positive and $x<a$.

a. Show that, putting $\lambda=2Iil/k$, the equation can be written as

$$\ddot{x}-\frac{k}{m}\frac{x^2-ax+\lambda}{a-x}=0$$

b. Put the equation in the frame work of Hamiltonian systems.
c. Copute the critical points and characterize them. Does the result agree with the Hamiltonian nature of the problem?
d. Skech the phase-plane for various value of λ

Jawab:

a. $m\ddot{x}+kx-\dfrac{2Iil}{a-x}=0$ ...... (*)

substitusikan $\lambda=\dfrac{2Iil}{k}$ ke persamaan (*)

$$m\ddot{x}+kx-\frac{\lambda k}{a-x}=0$$

$$\Leftrightarrow m\ddot{x}+\frac{k\ \ ax-x^2-\lambda}{a-x}=0$$

$$\Leftrightarrow \ddot{x}-\frac{k\ \ x^2-ax+\lambda}{m\ \ a-x}=0$$

$$\Leftrightarrow m\ddot{x}+\frac{a-x\ \ kx-\lambda x}{a-x}=0$$

$$\Leftrightarrow m\ddot{x}+\frac{akx-kx^2-\lambda x}{a-x}=0$$

b.

$$\dot{x}=y$$

$$\dot{y}=k\frac{x^2-ax+\lambda}{m\ \ a-x}$$

$$\frac{dx}{dy}=\frac{y}{k\left(\frac{x^2-ax+\lambda}{m\ \ a-x}\right)}$$

$$\int k\left(\frac{x^2-ax+\lambda}{m\ \ a-x}\right)dx=\int ydy$$

$$\int k\left(\frac{x^2-ax}{m\ \ a-x}+\frac{\lambda}{m\ \ a-x}\right)dx=\int ydy$$

$$\int k\left(\frac{x\ \ a-x}{m\ \ a-x}+\frac{\lambda}{m\ \ a-x}\right)dx=\int ydy$$

$x\neq a$

$$\Leftrightarrow \frac{k}{m}\frac{1}{2}x^2+k\frac{\lambda}{m}\ \ -\ln\ \ a-x\ \ =\frac{1}{2}x^2+C$$

$$\Leftrightarrow -\frac{k}{m}\frac{1}{2}x^2-k\frac{\lambda}{m}\ln\ \ a-x\ \ =\frac{1}{2}x^2+C$$

$F\ \ x,y\ \ =\frac{k}{m}\frac{1}{2}x^2+k\frac{\lambda}{m}\ln\ \ a-x\ \ +\frac{1}{2}y^2$: first integral

Dengan memasukan $\begin{matrix} p = m\dot{x} \\ q = x \end{matrix}$

sehingga hamiltoniannya

$$H(p,q) = \frac{1}{2}\frac{ka^2}{m} + k\frac{\lambda}{m}\ln(a-q) + \frac{1}{2}\frac{p^2}{m}$$

karena :

$$\dot{p} = -\frac{\partial H}{\partial q} = -\left(\frac{kq}{m} + \frac{kx}{m}.\frac{1}{a-q} - 1\right) = -\frac{kq}{m} - \frac{k\lambda}{m(a-q)}$$

$$\dot{q} = -\frac{\partial H}{\partial p} = \frac{p}{m}$$

c.

$$\dot{x} = y$$

$$\dot{y} = k\frac{x^2 - ax + \lambda}{m(a-x)}$$

Titik kritis diperoleh dari

$$\dot{x} = 0$$

$$\dot{y} = 0 \quad \rightarrow \quad k\frac{x^2 - ax + \lambda}{m(a-x)} = 0$$

$$k(x^2 - ax + \lambda) = 0$$

karena k bilangan positif maka $(x^2 - ax + \lambda) = 0$

jadi $x_{1,2} = \dfrac{a \pm \sqrt{a^2 - 4\lambda}}{2}$

Titik kritis tergantung dari nilai $a^2 - 4\lambda$

1) Jika

$$a^2 - 4\lambda > 0 \text{ maka } a^2 > 4\lambda$$

$$\Leftrightarrow \lambda < \frac{a^2}{4}$$

Jadi untuk $\lambda < \dfrac{a^2}{4}$ terdapat 2 titik kritis yaitu $(x_1, 0)$ dan $(x_2, 0)$

$$x_1 = \frac{a + \sqrt{a^2 - 4\lambda}}{2} \text{ dan } x_2 = \frac{a - \sqrt{a^2 - 4\lambda}}{2}$$

$x_1, 0$ membentuk saddle dan $x_2, 0$ membentuk centre

2) Jika

$$a^2 - 4\lambda = 0 \text{ maka } a^2 = 4\lambda$$

$$\Leftrightarrow \lambda = \frac{a^2}{4}$$

Jadi untuk $\lambda = \frac{a^2}{4}$ terdapat 2 titik kritis yang bergabung menjadi titik kritis degenerate.

3) Jika

$$a^2 - 4\lambda < 0 \text{ maka } a^2 < 4\lambda$$

$$\Leftrightarrow \lambda > \frac{a^2}{4}$$

Jadi untuk $\lambda > \frac{a^2}{4}$ tidak ada titik kritis sehingga tidak ada titik setimbang.

d) Sketsa

Problem 2-5

Find the critical points of the system

$$\dot{x} = y$$

$$\dot{y} = x - 2x^3$$

Characterize them the critical points by linier analysis and determine their attraction properties.

Penyelesaian:

Misalkan :

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 \end{aligned} \qquad (2.5.1)$$

Titik kritis dari sistem (2.5.1) diperoleh dari persamaan :

$$\dot{x} = 0 \Leftrightarrow y = 0$$

$$\dot{y} = 0 \Leftrightarrow x - 2x^3 = 0$$

sehingga diperoleh titik kritis : $(0,0), (\pm \tfrac{1}{2}\sqrt{2}, 0)$.

Proses linierisasi dipersekitaran titik kritis akan menghasilkan bentuk :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x_1} & \dfrac{\partial \dot{x}}{\partial x_2} \\ \dfrac{\partial \dot{y}}{\partial x_1} & \dfrac{\partial \dot{y}}{\partial x_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \cdots \qquad (2.5.2)$$

Misalkan $J = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x_1} & \dfrac{\partial \dot{x}}{\partial x_2} \\ \dfrac{\partial \dot{y}}{\partial x_1} & \dfrac{\partial \dot{y}}{\partial x_2} \end{pmatrix}$, maka $J = \begin{pmatrix} 0 & 1 \\ 1 - 6x^2 & 0 \end{pmatrix}$

dan persamaan karakteristik dari (2.5.2) diberikan oleh :

$$|\lambda I - J| = 0 \qquad (2.5.3)$$

Uji titik kritis:

- Di titik (0,0), diperoleh :

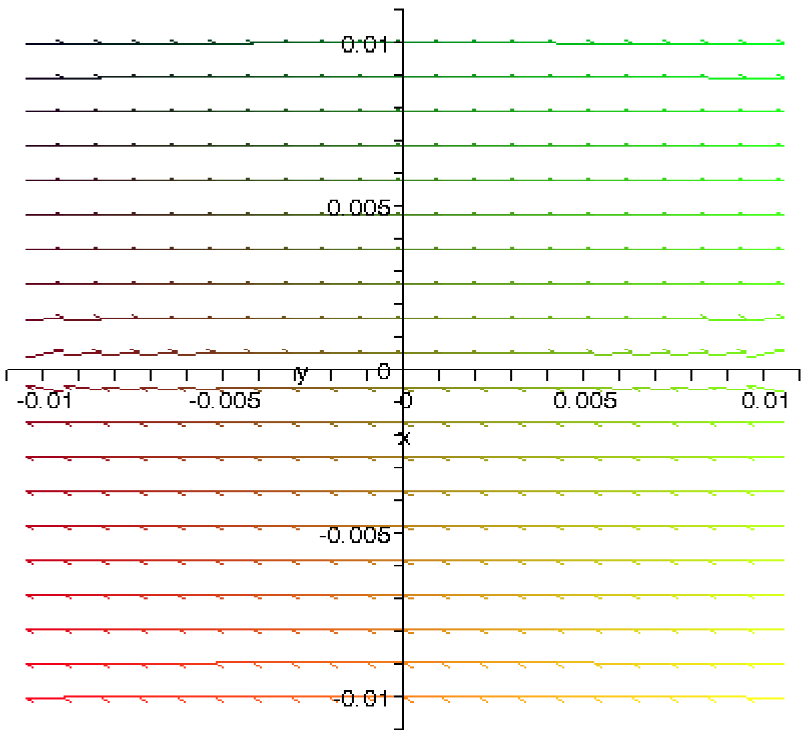
$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

sehingga diperoleh persamaan karakteristik :

$$\lambda^2 - 1 = 0$$

Jadi diperoleh nilai eigen $\lambda = \pm 1$.

Oleh karena di titik (0,0) sistem (2.5.1) mempunyai dua nilai eigen real yang berbeda tanda, berarti titik kritis (0,0) merupakan *saddle point*.



- Di titik $(\pm \frac{1}{2}\sqrt{2}, 0)$, diperoleh :

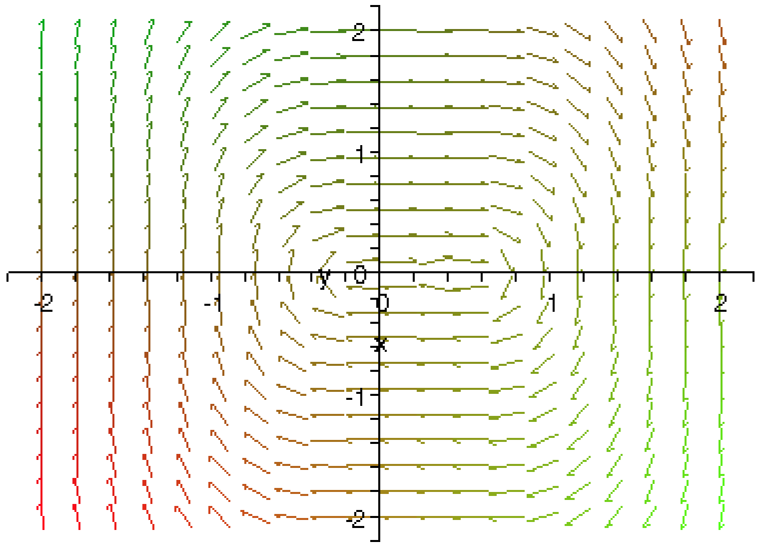
$$J_{(\pm \frac{1}{2}\sqrt{2},0)} = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$$

sehingga diperoleh persamaan karakteristik :

$$\lambda^2 + 2 = 0$$

Jadi diperoleh nilai eigen $\lambda = \pm\sqrt{2}\ i$.

Oleh karena di titik $(\pm \frac{1}{2}\sqrt{2}, 0)$ sistem (2.5.1) mempunyai dua nilai eigen imajiner murni, berarti titik kritis $(\pm \frac{1}{2}\sqrt{2}, 0)$ berupa centre.



2
1
y
0
-2
-1
0
1
2
x
-1
-2

Problem 2-6

Consider the system

$$\dot{x} = x$$

$$\dot{y} = y$$

a. find the first integral
b. Can we derive the equation from a Hamiltonian function?

Penyelesaian:

Pandang $F : R^2 \to R$

Agar *F* menjadi sebuah integral pertama, maka nilai F harus konstan sepanjang solusi.

$$\frac{d}{dt} F(x(t), y(t)) = 0$$

$$\Leftrightarrow \frac{\partial F}{\partial x} x + \frac{\partial F}{\partial y} y = 0$$

$$\Leftrightarrow \frac{\partial F}{\partial x} x = -\frac{\partial F}{\partial y} y$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$\Leftrightarrow \frac{1}{y} dy = \frac{1}{x} dx$$

$$\Leftrightarrow \ln y - \ln x = c$$

$$\Leftrightarrow \ln \frac{y}{x} = c$$

$$\Leftrightarrow \frac{y}{x} = e^c = k$$

Sehingga $F(x, y) = \frac{y}{x}$

Karena $div(\overline{f}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1 + 1 = 2 \neq 0$ maka *vector field*-nya tidak *volume preserving*, akibatnya system persamaan tersebut bukan *Hamiltonian System.*

Problem 2-7

Consider in R^n the system $\dot{x} = f(x)$ with divergence $\nabla . f(x) = 0$. So the phase flow is volume preserving. Does this mean that a first integral exist; solve for *n=2*

Penyelesaian:

misal $\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$ ...(*)

fase flow persamaan (*) ditentukan oleh

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

$$\Leftrightarrow f(x, y)\frac{dy}{dx} - g(x, y) = 0 \qquad ...(**)$$

jika persamaan (**) eksak maka

$$\frac{\partial f(x, y)}{\partial x} = -\frac{\partial g(x, y)}{\partial y}$$

Akibatnya terdapat $F(x, y) = c$ yang memenuhi persamaan (**).

Persamaan $\frac{\partial f(x, y)}{\partial x} = -\frac{\partial g(x, y)}{\partial y}$ ekivalen dengan $\frac{\partial f(x, y)}{\partial x} + \frac{\partial g(x, y)}{\partial y} = 0 \Leftrightarrow \nabla . \overline{f} = 0$.

Sehingga $\dot{x} = f(x)$ di R^n dengan $\nabla f(x) = 0$ integral pertamanya akan ada jika persamaan untuk phase flow-nya membentuk persamaan diferensial eksak.

Problem 2-8

Determine the critical points of the system

$$\dot{x} = x^2 - y^3$$

$$\dot{y} = 2x(x^2 - y)$$

Are there attractor in the system? Determine the first integral. Do the periodic solutions exist?

Penyelesaian:

Akan ditentukan integral pertama dari system persamaan diferensial di atas.

$$\frac{dy}{dx} = \frac{2x^3 - 2xy}{x^2 - y^3}$$

$$(x^2 - y^3)dy - (2x^3 - 2xy)dx = 0$$

Misal $M(x, y) =$ $(x^2 - y^3)$ dan $N(x, y) =$ $-(2x^3 - 2xy)$

Karena $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y} = 2x$

Persamaan tersebut merupakan persamaan diferensial eksak, sehingga ada $F(x, y) = k$ yang memenuhi $\frac{\partial F(x,y)}{\partial y} = M(x,y)$ dan $F : R^2 \to R$.

$$F(x,y) = \int (2xy - 2x^3)dx$$

$$= x^2 y - \frac{1}{2}x^4 + c(y)$$

$$\frac{\partial F(x,y)}{\partial y} = x^2 - c`(y)$$

Sedangkan $\frac{\partial F(x,y)}{\partial x} = x^2 - y^3$ sehingga $c`(y) = - y^3$

$$c(y) = -\frac{1}{4}y^4 + k$$

$\therefore F(x,y) = x^2 y - \frac{1}{2}x^4 - \frac{1}{4}y^4 + k$ adalah bentuk integral pertamanya.