Linear Algebra

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Outline

The general form

The system of m linear equations with n variables $x_1, ..., x_n$ has general form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n = &\quad b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = &\quad b_2 \\ \vdots &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = &\quad b_m \end{aligned}$$

Matrix form

The system of m linear equations with n variables $x_1, ..., x_n$ can be written as

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Note: A:= Coefficient matrix
x:= Variable matrix
b:= Constant matrix

Some terminologies

1. A vector $y \in \mathbb{R}^n$ is a solution of the system $Ax = b$ if $Ay = b$ holds
2. A system $Ax = b$ that has solution is said to be consistent

Example

Consider the system

$$2x_1 + 5x_2 = 8$$

$$3x_1 - 2x_2 = -7.$$

This system has unique solution $x = (-1, 2)$.

Example

Consider the system

$$x_1 + 2x_2 = 4$$

$$3x_1 + 6x_2 = 12.$$

(0,2) and (4,0) are the solutions of this system.
More over, $(4-2t, t)$, $t \in \mathbb{R}$ is general solution of this system.

Example

Consider the system

$$2x_1 + x_2 = 2$$
$$-4x_1 - 2x_2 = 0.$$

This system has no solution.

Fact

There are three possibilities regarding the existence of the solution, namely:

1. $Ax = b$ has unique solution
2. $Ax = b$ has many solutions
3. $Ax = b$ has no solution

Elementary Row Operations

There are three row operations, namely:

1. Exchange two rows
2. multiplicating a row by nonzero scalar k
3. multiplicating a row by scalar k and then adding it to the other row

Theorem

Consider the system $Ax = b$.
The row operation does not change the solution of $Ax = b$.

Homogen SLE

A homogen system has form

$$Ax = 0$$

Theorem

Every homogen system has solution

Definition

A vector space X over $\mathbb{K}$ is a set X together with an addition

$$u + v, \;\; u, v \in X$$

and a scalar multiplication

$$\alpha u, \;\; \alpha \in \mathbb{K}, \;\; u \in X$$

where the following axioms hold:

V1. $u + v = v + u$

V2. $(u + v) + w = u + (v + w)$

V3. there exists $0 \in X$ such that $0 + u = u$.

V4. there exists $-x \in X$ such that $x + (-x) = 0$.

V5. $(\alpha + \beta)x = \alpha x + \beta x$

V6. $\alpha(u + v) = \alpha u + \alpha v$

V7. $(\alpha\beta)x = \alpha(\beta x)$

V8. $1x = x$

Example

Let $X = \mathbb{R}^n$ where $n = 1, 2, 3, \ldots$; that is, the set X consists of all the n-tuples

$$x = (x_1, \ldots, x_n) \text{ with} x_i \in \mathbb{R} \text{ for all } i$$

Define

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

Example

Let $X = P_2$; that is, the set X consists of all the polynomial with the degree at most 2.

$$P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0,\ a_1,\ a_2 \in \mathbb{R}\}$$

Define

$$(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

$$\alpha(a_0 + a_1 x + a_2 x^2) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2$$

Example

Let $X = M_{m\times n}(\mathbb{R})$; that is, the set X consists of all $m \times n$ matrices.
Define

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \dots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \dots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

$$\alpha \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \dots & \alpha a_{1n} \\ \vdots & \dots & \vdots \\ \alpha a_{m1} & \dots & \alpha a_{mn} \end{bmatrix}$$

Subspace

Let X be a vector space over $\mathbb{R}$, and $Y \subseteq X$, $(Y \neq \emptyset)$.
Y is a subspace of X if Y is itself a vector space over $\mathbb{R}$ with respect to the operations of vector addition and scalar multiplication X.

Example

Let $X = \mathbb{R}^2$.
Consider the following subsets of X:

1. $Y_1 = \{(x,0) \mid x \in \mathbb{R}\}$
2. $Y_2 = \{(x,2x) \mid x \in \mathbb{R}\}$

Theorem

Let X be a vector space over $\mathbb{R}$, $Y \subseteq X$, $(Y \neq \emptyset)$.
Y is a subspace of X if Y satisfies two following conditions:

1. $x + y \in Y, \ \forall x, y \in Y$
2. $\alpha x \in Y, \ \forall \alpha \in \mathbb{R}, \ x \in Y$

Definition-Linear Combination

Let X be a vector space over $\mathbb{R}$, and $S = \{x_1, ..., x_n\}$ are set of vectors in X. x is called linear combination of $S = \{x_1, ..., x_n\}$ if there exists scalars $\alpha_1, ..., \alpha_n$ such that

$$x = \alpha_1 x_1 + ... + \alpha_n x_n$$

Example

(4,5) is linear combination of (2,1) and (3,3), because we can write

$$-1(2,1)+2(3,3)=(4,5)$$

Definition-Spanning Set

Let X be a vector space over $\mathbb{R}$, and $S \subseteq X$.
X is spanned by S if every vector y in X is linear combination of S.

Example

Consider $S = \{(1,1),(1,2)\} \subset X = \mathbb{R}^2$.
If we take $(a,b) \in X$ arbitrarily, we can find scalars α, β such that

$$\alpha(1,1) + \beta(1,2) = (a,b).$$

Example

Consider the set $S = \{(1,0,0),(0,1,0),(0,0,1)\} \subset X = \mathbb{R}^3$. For every $(a,b,c) \in X$, we can write

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1).$$

So we conclude that S spans X.

Definition-linearly independent

Let $S = \{x_1, ..., x_n\}$ are set of vectors in X. S is linearly independent if

$$0 = \alpha_1 x_1 + ... + \alpha_n x_n$$

has unique solution.

Definition-Basis & Dimension

Let $S = \{x_1, ..., x_n\}$ are set of vectors in X.
S is called basis of X if S linearly independent and Spans X.
The number n of all vectors in the basis is called dimension of X. (dim(X)=n).

Definition

Example

Theorem

Definition

Let X be a vector space over $\mathbb{R}$. An inner product $\langle .,. \rangle$ is a function on $X \times X$ which satisfies:

1. $\langle x, x \rangle \geq 0$; and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
2. $\langle x, y \rangle = \langle y, x \rangle$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

The pair $(\langle .,. \rangle, X)$ is called inner product space.

Norm

Let $(\langle .,. \rangle, X)$ be an inner product space, and $x \in X$.
Norm (length) of x is the number

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

Angle

Let x, y are nonzero vectors in X. The angle between x and y is θ for which

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|.\|y\|}$$

C-S Inequality

$$|\langle x, y\rangle| \leq \|x\| \|y\|$$

Orthogonal Set

Let $S = \{x_1, ..., x_n\}$ are set of vectors in X.
S is an *orthogonal set* if $x_i \neq x_j$ when $i \neq j$.

Orthonormal Set

Let $S = \{x_1, ..., x_n\}$ are set of vectors in X.
S is an $orthonormal\ set$ if S is orthogonal and $\|x_i\| = 1, \forall i$

Projection

Suppose $S = \{x_1, ..., x_r\}$ is orthonormal basis for subspace W of X, and $x \in X$. The projection of x along W is

$$Proj_W x = \langle x, x_1 \rangle x_1 + ... + \langle x, x_r \rangle x_r$$

Gram-Schmidt Process

Definition

Let V, W are vector spaces over $\mathbb{R}$.
The transformation

$$f : V \to W$$

is said to be linear if for all $x, y \in V, \alpha \in \mathbb{R}$, the following axioms hold:

1. $f(x + y) = f(x) + f(y)$
2. $f(\alpha x) = \alpha f(x)$

Definition

Let $f : V \to W$ be a linear transformation.
We define Kernel and Range of f as sets

$$Ker(f) = \{v \in V \mid f(v) = 0\}$$

and

$$R(f) = \{w \in W \mid w = f(u),\ u \in V\}$$

Theorem

Let $f : V \to W$ be a linear

1. $Ker(f)$ is a subspace of V.
2. $R(f)$ is a subspace of W

Representation Matrix

Let V, W be finite dimensional vector spaces with basis $B = \{v_1, ..., v_n\}$ for V, and $B' = \{w_1, ..., w_n\}$ for W. If $f : V \to W$ is linear, representation matrix of f is defined by

$$[f]_{BB'} = \begin{bmatrix} [f(v_1)]_{B'} & \dots & [f(v_n)]_{B'} \end{bmatrix}$$

Definition

Let M be a $n \times n$ matrix over $\mathbb{R}$. $\lambda \in \mathbb{R}$ is called an eigenvalue of M if there exists a nonzero vector $x \in \mathbb{R}^n$ for which

$$Mx = \lambda x$$

Every vector satisfying this relation is then called an eigenvector of M belonging to the eigenvalue λ.

Fact

Let λ be an eigenvalue of M, and x is the correspondence eigenvector. Consider the relation

$$Mx = \lambda x.$$

This equation is equivalent to

$$(\lambda I - M)x = 0.$$

Then x is a solution of the system $(\lambda I - M)x = 0$.
Therefore, we have

$$|(\lambda I - M)| = 0.$$

Theorem

The following are equivalent:

1. λ is an eigenvalue of M
2. x is a solution of the system $(\lambda I - M)x = 0$.
3. $|(\lambda I - M)| = 0$

Definition

A $n \times n$ matrix M is said to be diagonalizable if there exists a nonsingular matrix P for which

$$D = P^{-1}MP$$

where D is a diagonal matrix.

Theorem

Let M be a $n \times n$ matrix over $\mathbb{R}$. If M has linearly independent set of n eigenvectors then M is diagonalizable. Moreover,

Theorem

Let M be a $n \times n$ matrix over $\mathbb{R}$. If M has n distinct eigenvalues then M is diagonalizable.

Theorem

Theorem