The Integral

chapter 6

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- The Indefinite Integral
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Antiderivative

An *antiderivative* of a function f is a function F such that

$$F' = f$$

Ex. An *antiderivative* of $f(x) = 6x$ is $F(x) = 3x^2 + 2$ since $F'(x) = f(x)$.

Indefinite Integral

The expression: $\int f(x)dx$

read "the indefinite integral of *f* with respect to *x*," means to find the set of all antiderivatives of *f*.

$$\int f(x)dx$$

x is called the variable of integration

Integral sign

Integrand

Constant of Integration

Every antiderivative *F* of *f* must be of the form *F*(*x*) = *G*(*x*) + *C*, where *C* is a constant.

Notice $\int 6x\,dx = 3x^2 + C$

Represents every possible antiderivative of 6*x*.

Power Rule for the Indefinite Integral, Part I

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$$

Ex. $$\int x^3 dx = \frac{x^4}{4} + C$$

Power Rule for the Indefinite Integral, Part II

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

Indefinite Integral of e^x and b^x

$$\int e^x dx = e^x + C$$

$$\int b^x dx = \frac{b^x}{\ln b} + C$$

Sum and Difference Rules

$$\int \left(f \pm g \right) dx = \int f dx \pm \int g dx$$

Ex. $\int \left(x^2 + x \right) dx = \int x^2 dx + \int x dx = \frac{x^3}{3} + \frac{x^2}{2} + C$

Constant Multiple Rule

$$\int k f(x) dx = k \int f(x) dx \quad (k \text{ constant})$$

Ex. $\int 2x^3 dx = 2 \int x^3 dx = 2 \frac{x^4}{4} + C = \frac{x^4}{2} + C$

Integral Example/Different Variable

Ex. Find the indefinite integral:

$$\int \left(3e^{u} - \frac{7}{u} + 2u^{2} - 6 \right) du$$

$$= 3\int e^{u} du - 7\int \frac{1}{u} du + 2\int u^{2} du - \int 6 du$$

$$= 3e^{u} - 7\ln\left|u\right| = \frac{2}{3}u^{3} - 6u + C$$

Position, Velocity, and Acceleration

Derivative Form

If $s = s(t)$ is the position function of an object at time t, then

Velocity $= v = \dfrac{ds}{dt}$ Acceleration $= a = \dfrac{dv}{dt}$

Integral Form

$$s(t) = \int v(t)\,dt \qquad v(t) = \int a(t)\,dt$$

Integration by Substitution

Method of integration related to chain rule differentiation. If *u* is a function of *x*, then we can use the formula

$$\int f\,dx = \int \left(\frac{f}{du/dx} \right) du$$

Integration by Substitution

Ex. Consider the integral: $\int 3x^2 \left(x^3 - 5 \right)^9 dx$

pick $u = x^3 + 5$, then $du = 3x^2 dx$

$$\frac{du}{3x^2} = dx$$

$$\int u^9 du \quad = \frac{u^{10}}{10} + C \quad = \frac{\left(x^3 + 5 \right)^{10}}{10} + C$$

Sub to get | Integrate | Back Substitute

Ex. Evaluate $\int x\sqrt{5x^2-7}\,dx$

Let $u = 5x^2 - 7$ then $\frac{du}{10x} = dx$

Pick u, compute du

$$\int x\sqrt{5x^2-7}\,dx = \int \frac{1}{10} u^{1/2}\,du$$

Sub in

$$= \left(\frac{1}{10}\right)\frac{u^{3/2}}{3/2} + C$$

Integrate

$$= \frac{{5x^2-7}^{\;3/2}}{15} + C$$

Sub in

Ex. Evaluate $\int \frac{dx}{x\,(\ln x)^{3}}$

Let $u = \ln x$ then $x\,du = dx$

$$\int \frac{dx}{x\,(\ln x)^{3}} = \int u^{-3}\,du$$

$$= \frac{u^{-2}}{-2} + C$$

$$= \frac{(\ln x)^{-2}}{-2} + C$$

Ex. Evaluate $\displaystyle\int \frac{e^{3t}\,dt}{e^{3t}+2}$

Let $u = e^{3t} + 2$ then $\dfrac{du}{3e^{3t}} = dt$

$$\int \frac{e^{3t}\,dt}{e^{3t}+2} = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{\ln|u|}{3} + C$$

$$= \frac{\ln\ e^{3t}+2}{3} + C$$

Shortcuts: Integrals of Expressions Involving $ax + b$

Rule

$$\int ax+b^{\;n}\,dx=\frac{ax+b^{\;n+1}}{a(n+1)}+C \qquad n\neq -1$$

$$\int ax+b^{\;-1}\,dx=\frac{1}{a}\ln\left|ax+b\right|+C$$

$$\int e^{ax+b}dx=\frac{1}{a}e^{ax+b}+C$$

$$\int c^{ax+b}dx=\frac{1}{a\ln c}c^{ax+b}+C$$

Riemann Sum

If f is a continuous function, then the left Riemann sum with n equal subdivisions for f over the interval $[a, b]$ is defined to be

$$\sum_{k=0}^{n-1} f\left(x_k\right)\Delta x$$

$$= f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

$$= \left[f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right]\Delta x$$

where $a = x_0 < x_1 < \ldots < x_n = b$ are the subdivisions and $\Delta x = (b - a)/n$.

The Definite Integral

If f is a continuous function, the definite integral of f from a to b is defined to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(x_k\right) \Delta x$$

The function f is called the integrand, the numbers a and b are called the limits of integration, and the variable x is called the variable of integration.

Approximating the Definite Integral

Ex. Calculate the Riemann sum for the integral $\int_0^2 x^2 dx$ using $n = 10$.

$$\sum_{k=0}^{n-1} f\left(x_k\right)\Delta x = \sum_{k=0}^{9} x_k{}^2\left(\frac{1}{5}\right)$$

$$=\left[(1/5)^2 + (2/5)^2 + \ldots + (9/5)^2\right](1/5)$$

$$\boxed{=2.28}$$

The Definite Integral

$$\int_a^b f(x)\,dx$$

is read "the integral, from a to b of $f(x)dx$."

Also note that the variable x is a "dummy variable."

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

The Definite Integral As a Total

If $r(x)$ is the rate of change of a quantity Q (in units of Q per unit of x), then the total or accumulated change of the quantity as x changes from a to b is given by

$$\text{Total change in quantity } Q = \int_a^b r(x)\,dx$$

The Definite Integral As a Total

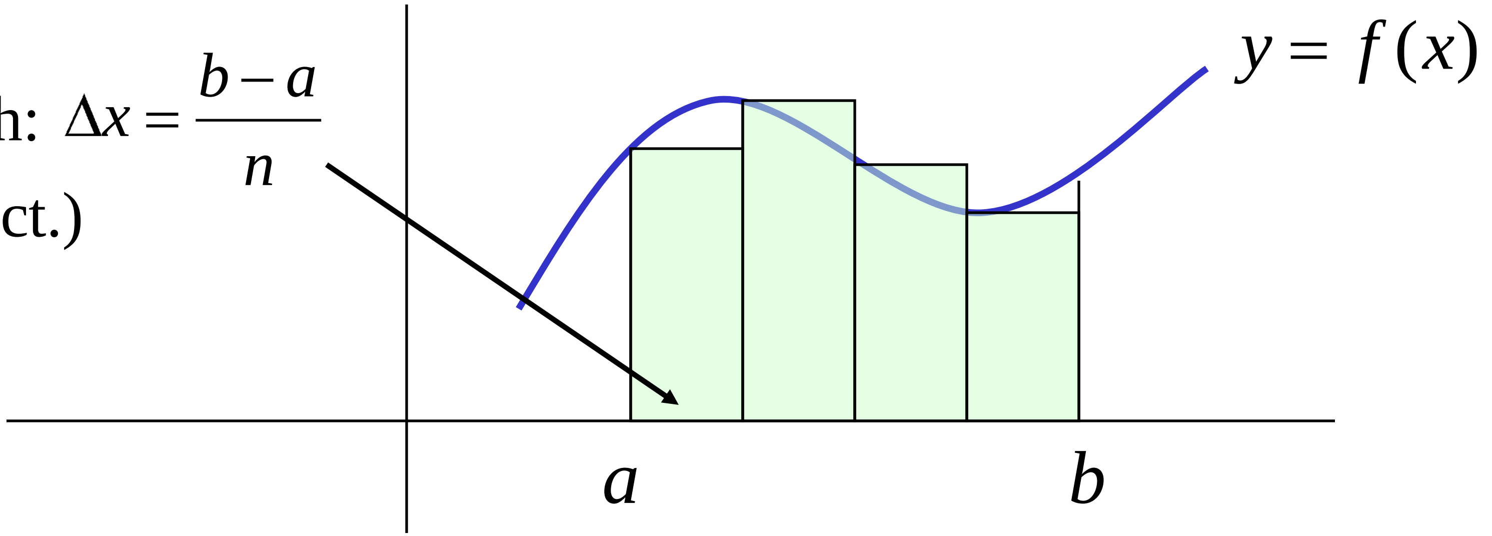
Ex. If at time *t* minutes you are traveling at a rate of *v*(*t*) feet per minute, then the total distance traveled in feet from minute 2 to minute 10 is given by

$$\text{Total change in distance} = \int_{2}^{10} v(t)\,dt$$

Area Under a Graph

Width: $\Delta x = \frac{b-a}{n}$

(n rect.)



Idea: To find the exact area under the graph of a function.

Method: Use an infinite number of rectangles of equal width and compute their area with a limit.

Approximating Area

Approximate the area under the graph of

$$f(x) = 2x^2 \quad \text{on} \quad 0, 2$$

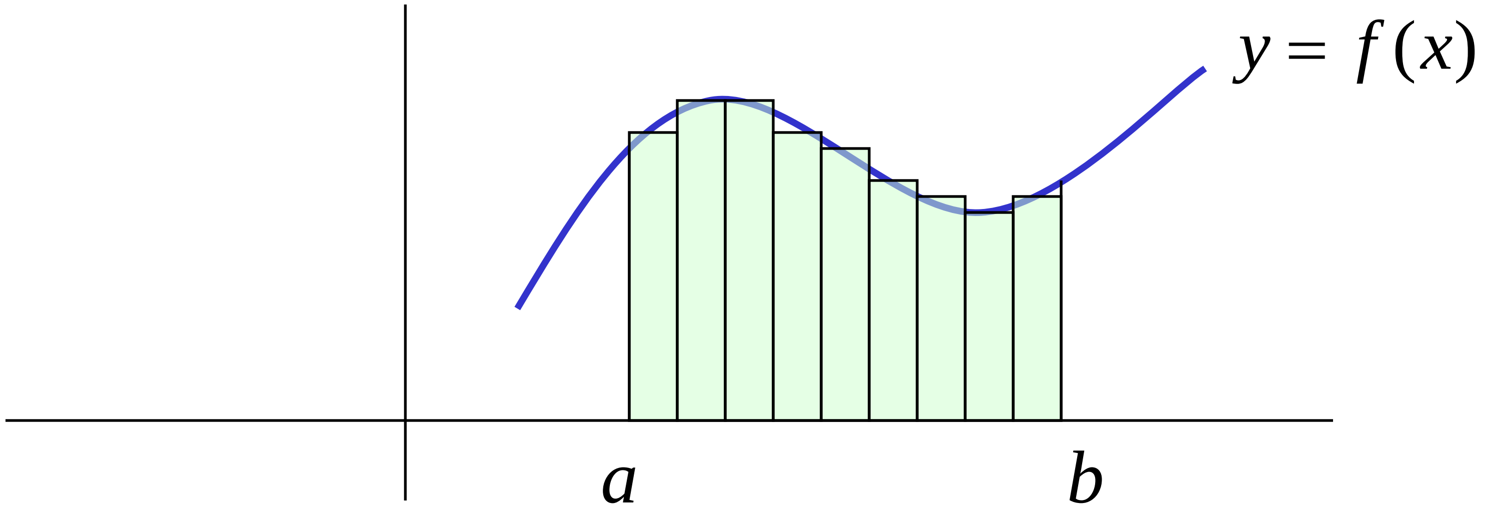
using $n = 4$.

$$A \approx \Delta x \;\; f(x_0) + f(x_1) + f(x_2) + f(x_3)$$

$$A \approx \frac{1}{2}\left[f \;\; 0 \;\; + f\left(\frac{1}{2}\right) + f \;\; 1 \;\; + f\left(\frac{3}{2}\right)\right]$$

$$A \approx \frac{1}{2}\left[0 + \frac{1}{2} + 2 + \frac{9}{2}\right] = \frac{7}{2}$$

Area Under a Graph

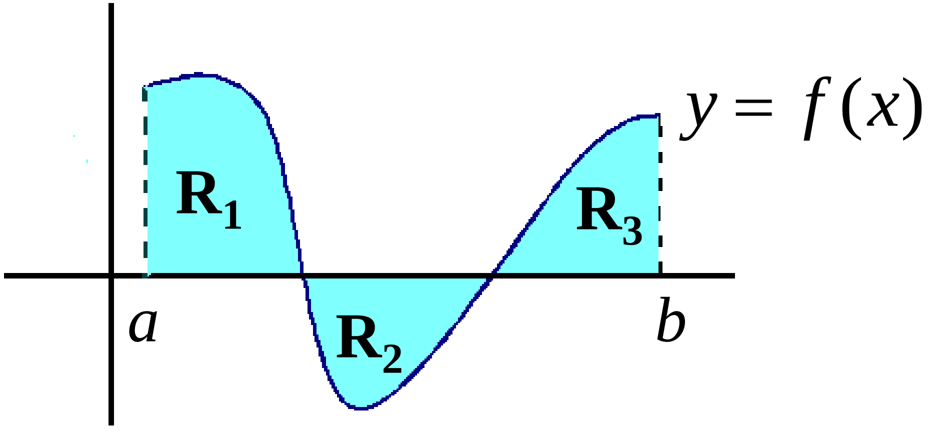


f continuous, nonnegative on $[a, b]$. The area is

$$\text{Area} = \lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(x_k\right) \Delta x$$

$$= \int_a^b f(x)\,dx$$

Geometric Interpretation (All Functions)

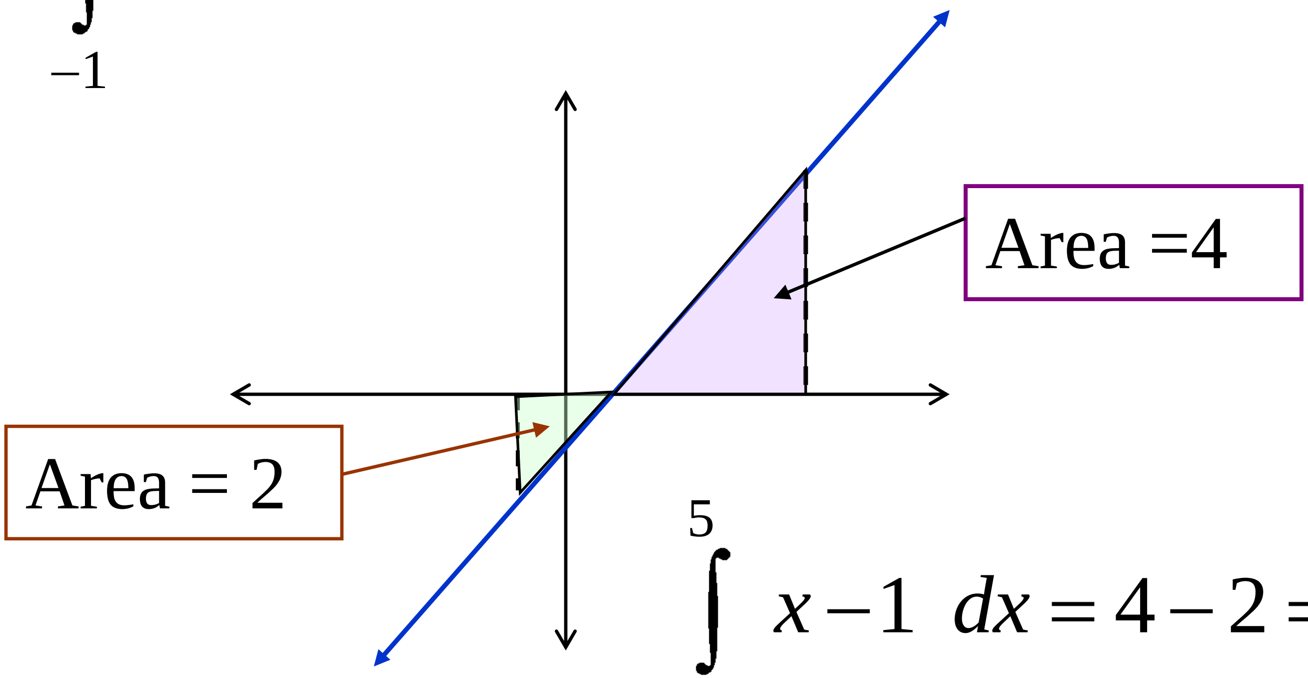


$$\int_a^b f(x)dx = \text{Area of } R_1 - \text{Area of } R_2 + \text{Area of } R_3$$

Area Using Geometry

Ex. Use geometry to compute the integral

$$\int_{-1}^{5} x-1 \; dx$$



$$\int_{-1}^{5} x-1 \; dx = 4-2 = 2$$

Fundamental Theorem of Calculus

Let f be a continuous function on $[a, b]$.

1. If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.

2. If F is any continuous antiderivative of f and is defined on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus

Ex. If $A(x)=\int_a^x \sqrt[3]{t^4+5t}\,dt$, find $A'(x)$.

$$A'(x)=\sqrt[3]{x^4+5x}$$

Evaluating the Definite Integral

Ex. Calculate $\int_1^5 \left(2x - \frac{1}{x} + 1 \right) dx$

$$\int_1^5 \left(2x - \frac{1}{x} + 1 \right) dx = \left. x^2 - \ln x + x \right|_1^5$$

$$= 5^2 - \ln 5 + 5 - 1^2 - \ln 1 + 1$$

$$= 28 - \ln 5 \approx 26.39056$$

Substitution for Definite Integrals

Ex. Calculate $\int_0^1 2x\ \ x^2+3\ {}^{1/2}\, dx$

let $u = x^2 + 3x$

then $\dfrac{du}{2x} = dx$

Notice limits change

$$\int_0^1 2x\ \ x^2+3x\ {}^{1/2}\, dx = \int_0^4 u^{1/2}\, du$$

$$= \frac{2}{3} u^{3/2} \Big|_0^4 \quad = \frac{16}{3}$$

Computing Area

Ex. Find the area enclosed by the x-axis, the vertical lines $x = 0$, $x = 2$ and the graph of

$y = 2x^2$.

$$\int_0^2 2x^3 dx$$

Gives the area since $2x^3$ is nonnegative on [0, 2].

$$\int_0^2 2x^3 dx = \frac{1}{2}x^4\Big|_0^2 = \frac{1}{2}\,2^4 - \frac{1}{2}\,0^4 = 8$$

Antiderivative

Fund. Thm. of Calculus