# Chaotic Dynamics in an Autoparametric System with Parametric Excitation

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*We consider an autoparametric system, which consists of an oscillator, coupled with parametrically-excited subsystem. The oscillator and the subsystem are in 1:1 internal resonance. The excited subsystem is in 1:2 resonance with the external forcing. The method of averaging is used to yield a set of autonomous equation of the approximation to the response of the system. We find various types of bifurcation, leading to non-trivial periodic or quasi-periodic solutions. Using numerical bifurcation continuation , we found sequences of period-doublings, leading to chaotic solutions. To analyze the parameter range for which a Shilnikov type homoclinic orbit exists, we used global perturbation technique developed by Kovacic and Wiggins [1]. This orbit gives rise to a well-described chaotic dynamics. The theoretical results are found to be in a good agreement with the results obtained by simulation.*

Keywords: autoparametric system, chaotic dynamics, parametric Excitation

## INTRODUCTION

An autoparametric system is a vibrating system, which consists of at least two subsystems: the oscillator and the excited subsystem. The oscillator is coupled to the excited subsystem in a nonlinear way, but such that the excited subsystem can be at rest while the oscillator is vibrating. We call this state the semi-trivial solution. In physics it is called a normal mode.The classical example of an autoparametric system is the elastic pendulum, which consists of a spring fixed at one end. The spring may swing in a plane like a pendulum and oscillate at the same time. Until recently, research in an autoparametric system was concerned mainly with the system consisting of a vibrating single mass with an attached pendulum (Figure 1).

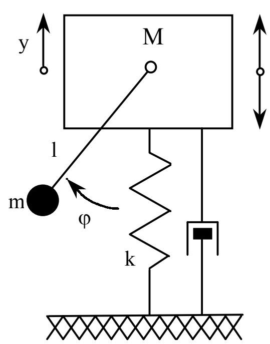


**FIGURE 1.** An autoparametric system consisting of a mass mount on a spring (the oscillator) and of a pendulum attached to the mass (the excited subsystem).

In the literature up till 1992, the excitation of the oscillator was considered to be external. In [2], It is shown that not only external excitation but also parametric or self-excitation of the oscillator can be the source of autoparametric excitation (see [3] and [4] for more backgrounds). In studying an autoparametric system, the determination of stability and instability conditions of the semi-trivial solution or normal mode is always the first step. After that we look for other periodic solutions, bifurcation and chaotic solutions. In this paper we shall consider an autoparametric system where the oscillator is excited parametrically with 1:2 resonance and that there exists an internal 1:1 resonance. The coupling of the oscillator and the excited subsystem are in cubic nonlinearities. Using numerical simulation and mathematical analysis, we will concern on study the chaotic solutions, which arise in this system.

## THE AVERAGED SYSTEMS

We consider an autoparametric system of the form:

$$\begin{aligned} x''+x+\varepsilon(\kappa_1 x'+\sigma_1 x+a\cos(2\tau)x+\frac{4}{3}x^3+c_1 y^2 x) &= 0 \\ y''+y+\varepsilon(\kappa_2 y'+\sigma_2 y+\frac{4}{3}y^3+c_2 y x^2) &= 0 \end{aligned} \qquad (1)$$

The first equation represents the oscillator and the second one is the excited subsystem. An accent, as in $x'$, will indicate differentiation with respect to time $\tau$ and $x, y \in \Re$. The damping coefficients $\kappa_1$ and $\kappa_2$ are in order one, so are the amplitude of forcing $a$ and the coefficient of couplings c1 and c2. The $\sigma_1$ and $\sigma_2$ are the detunings from exact resonance. Transform system (1) using the transformation

$$x = u_1\cos\tau + v_1\sin\tau \quad ; \quad x' = u_1\cos\tau + v_1\sin\tau$$
$$y = u_2\cos\tau + v_2\sin\tau \quad ; \quad y' = u_2\cos\tau + v_2\sin\tau$$

and averaging over $\tau$ then rescaling $\tau = \frac{\varepsilon}{2}\bar{\tau}$ will lead to system in the form

$$X' = F(X), \text{ where } X = (u_1\ v_1\ u_2\ v_2)^T \qquad (2)$$

see [5] for details. In the sequel a different formulation of system (2) will often be used transformation to action-angle variables

$$u_i = -\sqrt{2R_i}\cos\theta_i, \text{ and } v_i = \sqrt{2R_i}\sin\theta_i, \quad i = 1,2.$$

yields system of the form

$$Y' = G(Y), \text{ where } Y = (R_1\ \theta_1\ R_2\ \theta_2)^T \qquad (3)$$

The semi-trivial solution is found by taking $y = 0$. Note that the system (1) is invariant under $(x, y) \to (-x, y)$, $(x, y) \to (x, -y)$, and $(x, y) \to (-x, -y)$. In particular the first symmetry will be important in the analysis of this system.

## PERIODIC DOUBLING BIFURCATIONS AND CHAOTIC SOLUTION

On the stability boundary of the semi-trivial solution in system (2), it is easy to find an analytical condition when the semi-trivial solution can undergo a pitchfork bifurcation. We have used the bifurcation

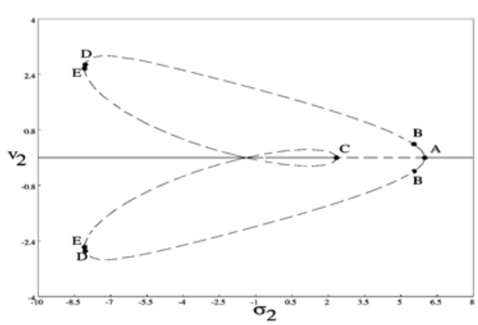


**FIGURE 2.** The stability diagram of system (2) in the $(\sigma_2, v_2)$-plane for fixed values $\kappa_1 = \kappa_2 = c_1 = 1$, $c_2 = -1$, $a = 2.1$, and $\sigma_1 = -8$.

Solid curves mean that solutions are stable and the dashed line that it is unstable. Continuation program CONTENT (Kutznetsov [6]) is used to study the non-trivial solutions branching from these bifurcation points. In Figure 2, the semi-trivial solution branches at point A, a stable non-trivial solution bifurcates and then this non-trivial solution undergoes a Hopf bifurcation at point B. We point out that a fixed point and a periodic solution of system (2) correspond to a periodic and quasi-periodic solution, respectively, in the original , time dependent system. A supercritical Hopf bifurcation occurs at point B for $\sigma_2 = 5.5327$ and at point D for $\sigma_2 = 8.051$. A saddle-node bifurcation occurs at point E for $\sigma_2 = 3.079'$. We find a stable periodic orbit for all values of $\sigma_2$ in the interval 5.4119 < $\sigma_2$ <5.5371. As $\sigma_2$ is decreased, period doubling of the stable periodic solution is observed. There is an infinite number of such period doubling, until the value $\sigma_2$=5.2505 is reached. The values of $\sigma_2$ with 5.2505 < $\sigma_2$ <5.3195 produce a strange attractor (Figure 3). We find that the Lyapunov spectra of system (2) corresponding to parameter values above are

$\lambda_1 = 0.8411$, $\lambda_2 = -0.3864$, $\lambda_3 = -0.1596$, and $\lambda_4 = -0.2858$. Since this system contain one positive Lyapunov exponent, we conclude that the strange attractor is chaotic, with the magnitude of the exponent reflecting the time scale on which system become unpredictable (Wolf, Swift, Swinney, and Vastano [7]).

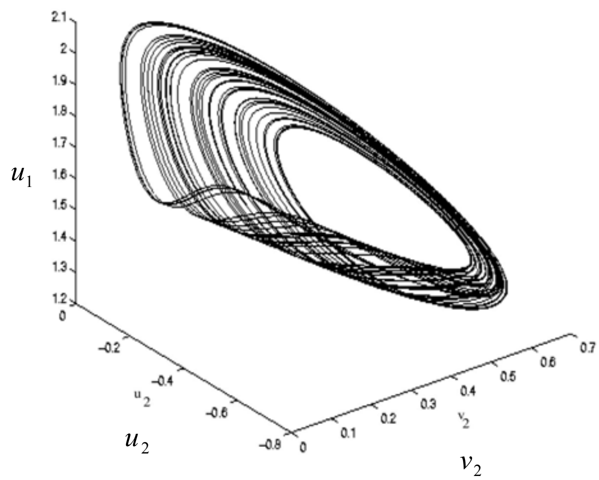


**FIGURE 3.** The strange attractor of the averaged system (2). The phase-portraits in the $(u_2, v_2, u_1)$ -space for $\kappa_1 = \kappa_2 = c_1 = 1$, $c_2 = -1$, $a = 2.1$, and $\sigma_1 = -8$ at the value $\sigma_2 = 5.3$. The Kaplan-Yorke dimension for $\sigma_2 = 5.3$ is 2.29.

## ANALYTICAL STUDY OF CHAOTIC SOLUTION BY USING GENERALIZED MELNIKOV METHOD

By using a generalized version of Melnikov's method , we find that for certain values of the parameters, the averaged system (3) has a homoclinic orbit of Shilnikov-type. The idea of this method is to study a system in which the unperturbed problem is an integrable Hamiltonian system having a normally hyperbolic invariant set whose stable and unstable manifold intersect nontransversally. The global geometry associated with the integrable structure is used to develop coordinates which are then used in determining if any of the homoclinic orbits to the normally hiperbolic invariant set survive under perturbation. To prove this involves an application of higher dimensional Melnikov theory developed by Kovacic and Wiggins (1992). A rather technical analysis shows the existence of a Shilnikov orbit in system (3), which implies chaotic dynamics, also for the original system.

In Figure 3, we show the area in parameter-plane where a Shilnikov orbit exits. It is bordered by lines $L_3$ and $L_4$. In that figure, we have also indicated the lines where the semi-trivial solution bifurcates to a non-trivial solution (lines $L_1$ and $L_5$), and where the Hopf–bifurcation of the non-trivial solution occurs ($L_2$). These last curves are found from previous analysis by a suitable rescaling. As we mentioned, the curves correspond to high degree with the numerical results obtained earlier. For example, for parameter-values $\kappa_1 = \kappa_2 = c_1 = 1,\ c_2 = -1,\ a = 2.1$, and $\sigma_1 = -8$. we numerically found a strange attractor for 5.2505< $\sigma_2$ <5.3195, whereas the values as predicted by the curves $L_3$ and $L_4$ yields 5.26< $\sigma_2$ <5.33.

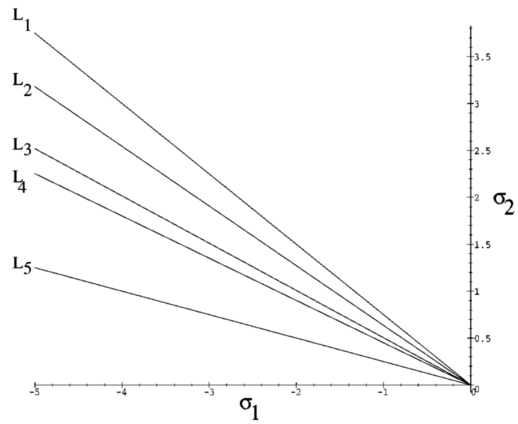


**FIGURE 4.** Parameter diagram of system (3) in the $(\sigma_1, \sigma_2)$-plane for values $\kappa_1 = \kappa_2 = c_1 = 1,\ c_2 = -1,\ a = 2.1$.

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