

BAB IV

FUNGSI μ REGULAR

KONSTRUKSI FUNGSI μ REGULAR

Proposisi IV.1

Hal. 55

Misalkan u fungsi panharmonik bernilai real pada Ω .

Maka secara lokal, $\exists f(z) = u(z) + iv(z)$ μ Regular

pada Ω . Dimana $v(x, y) = \int_a^y \left(u_x(x, t) - \mu u(x, t) \right) dt + \varphi(x)$

dan $\varphi(x) = e^{-\mu x} \left(\int e^{\mu x} u_y(x, a) dx + c \right)$

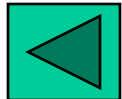


Akibat IV.1

Hal. 56

Misalkan u panharmonik bernilai real.

$$\left. \begin{array}{l} f_1 = u + iv_1 \text{ } \mu \text{ regular} \\ f_2 = u + iv_2 \text{ } \mu \text{ regular} \end{array} \right\} \Rightarrow v_1 - v_2 = ke^{-\mu x}, \exists k \in \mathfrak{R}$$

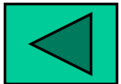


Akibat IV.2

Hal. 56

Misalkan f dan g fungsi μ regular.

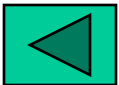
$$\left. \begin{array}{l} f = u + iv \\ g = u + iw \end{array} \right\} \Rightarrow f(z) = g(z) + ike^{-\mu x}, \exists k \in \mathfrak{R}.$$



Proposisi IV.2

Hal. 56

Misalkan v fungsi panharmonik bernilai real pada Ω .
 Maka secara lokal, $\exists f(z) = u(z) + iv(z)$ μ Regular
 pada Ω . Dimana $u(x, y) = -\int_a^y \left(v_x(x, t) + \mu v(x, t) \right) dt + \Phi(x)$
 dan $\Phi(x) = e^{\mu x} \left(e^{-\mu x} v_y(x, a) dx + c \right)$

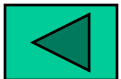


Akibat IV.3

Hal. 57

Misalkan v panharmonik bernilai real.

$$\left. \begin{array}{l} f_1 = u_1 + iv \text{ } \mu \text{ regular} \\ f_2 = u_2 + iv \text{ } \mu \text{ regular} \end{array} \right\} \Rightarrow u_1 - u_2 = ke^{\mu x}, \exists k \in \mathfrak{R}$$



Akibat IV.4
Hal. 57

Misalkan f dan g fungsi μ regular.

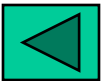
$$\left. \begin{array}{l} f = u + iv \\ g = w + iv \end{array} \right\} \Rightarrow f(z) = g(z) + ke^{\mu x}, \exists k \in \mathfrak{R}$$



Proposisi IV.3

Hal. 58

Fungsi μ regular konstan adalah fungsi nol



Proposisi IV.4,5,6

Hal. 58-59

Misalkan f fungsi μ regular pada Ω .

f bernilai real $\Rightarrow f = ke^{\mu x}, \exists k \in \mathbb{R}$.

f bernilai imajiner $\Rightarrow f(z) = ike^{-\mu x}, \exists k \in \mathbb{R}$

$\overline{f(z)}$ μ regular $\Rightarrow f(z) = Ae^{\mu x} + iBe^{-\mu x}, \exists A, B \in \mathbb{R}$



Pendefinisian Sekawan Panharmonik

Hal. 59

Fungsi v disebut sekawan panharmonik dengan u jika u dan v memenuhi persamaan Cauchy-Riemann yang diperumum (C-R-P), yaitu

$$u_x = v_y + \mu u$$

$$u_y = -v_x - \mu v$$



Akibat IV.5

Hal 59

v sekawan panharmonik dengan u



$-v$ sekawan panharmonik dengan $-u$



Proposisi IV.7

Hal. 59

Misalkan $f = u + iv$ fungsi μ Regular.

Fungsi u dan v saling sekawan panharmonik



$i \overline{f(z)}$ fungsi μ regular



Akibat IV.6

Hal. 60

Misalkan $f(z) = u(z) + iv(z)$ fungsi μ regular.

$i \overline{f(z)}$ fungsi μ regular



$$f(z) = A(1+i)e^{-\mu y} + B(1-i)e^{\mu y}, \exists A, B \in \mathbb{R}$$



Akibat IV.7

Hal. 60

Misalkan $f = u + iv$ fungsi μ regular.
 u dan v saling sekawan panharmonik



$$f(z) = A(1+i)e^{-\mu y} + B(1-i)e^{\mu y}, \exists A, B \in \mathbb{R}$$



Akibat IV.8

Hal. 60

Misalkan $f \neq 0$ regular.

Jika $if(z) \neq 0$ regular $\Rightarrow f$ fungsi nol



PENYAJIAN INTEGRAL FUNGSI μ REGULAR

Lihat [10]

Misalkan Γ sembarang lintasan tertutup sederhana pada Ω .
dan $f(z)$ fungsi μ regular pada Ω . Jika $r = |z - a|$ maka

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{a - z} \mu r K_0'(\mu r) dz + \frac{1}{2\pi i} \int_{\Gamma} f(z) K_0(\mu r) dz$$

asalkan a di dalam Γ . Jika a di luar Γ maka ruas kanan nol.



Lema IV.1
Hal. 61 (lihat [10])

Misalkan $h(z) \in C^1 \overline{\Omega}$.
 $\operatorname{Re} \oint (h(z)) = 0 \Rightarrow \operatorname{Im} \int_{\Gamma} h(z) dz = 0.$



Lema IV.2

Hal. 61

Misalkan $h(z) \in C^1 \Omega$.

$$\operatorname{Im} \int_{\Gamma} h(z) dz = 0 \Rightarrow \operatorname{Re} \int_{\Gamma} h(z) dz = 0.$$



Proposisi IV.8

Hal. 61

f dan g masing - masing fungsi μ regular pada Ω



$$\operatorname{Re} \int_{\Gamma} f(z)g(z)dz = 0$$



Akibat IV.9,11
Hal. 62

$$f \text{ } \mu \text{ regular pada } \Omega \Rightarrow * \operatorname{Re} \int_{\Gamma} f^2(z) dz = 0$$

$$* \int_{\Gamma} f^2(z) dz + \overline{\int_{\Gamma} f^2(z) dz} = 0$$



Akibat IV.10

Hal. 62

f dan g masing - masing fungsi μ regular pada Ω

$\Downarrow$

$$\int_{\Gamma} f(z)g(z)dz + \overline{\int_{\Gamma} f(z)g(z)dz} = 0$$



Akibat IV.12

Hal. 62

$$\left. \begin{array}{l} f(z) = u(z) + iv_1(z) \text{ fungsi } \mu \text{ regular} \\ g(z) = u(z) + iv_2(z) \text{ fungsi } \mu \text{ regular} \end{array} \right\} \Rightarrow \operatorname{Im} \int_{\Gamma} (f - g)(z) dz = 0$$



Akibat IV.13
Hal. 62

$$\left. \begin{array}{l} f(z) = u_1(z) + iv(z) \text{ fungsi } \mu \text{ regular} \\ g(z) = u_2(z) + iv(z) \text{ fungsi } \mu \text{ regular} \end{array} \right\} \Rightarrow \operatorname{Re} \int_{\Gamma} (f - g)(z) dz = 0$$



Lema IV.3,4,5

Hal. 63

Misalkan u panharmonik $\Rightarrow *f = \mu u + \overline{L}u$ μ regular (Duffin)

$$*f = \mu \bar{u} + \overline{L}u \quad \mu \text{ regular}$$

$$*f = i(\mu u - \mu \bar{u}) \quad \mu \text{ regular}$$



Proposisi IV.9

Hal.63

$$\left. \begin{array}{l} u \text{ panharmonik pada } \Omega \\ f \mu \text{ regular pada } \Omega \end{array} \right\} \Rightarrow \int_{\Gamma} f(z) \bar{L} u dz + \int_{\Gamma} u(z) L f(z) d\bar{z} = 0$$

untuk sembarang lintasan tertutup sederhana Γ dalam Ω

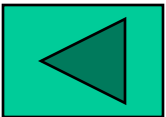


Lema IV.6,7
Hal.65

Misalkan $h(z) \in C^1(\Omega)$.

$$\operatorname{Re} \oint_{\Gamma} h(z) dz = 0 \Rightarrow \operatorname{Im} \int_{\Gamma} h(z) d\bar{z} = 0$$

$$\operatorname{Im} \oint_{\Gamma} h(z) dz = 0 \Rightarrow \operatorname{Re} \int_{\Gamma} h(z) d\bar{z} = 0$$



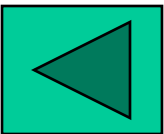
Proposisi IV.11

Hal. 66

f dan g masing - masing fungsi μ regular

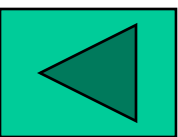


$$\operatorname{Re} \int_{\Gamma} \overline{f(z)g(z)} dz = 0$$



Akibat IV.14
Hal.66

$$f \text{ } \mu \text{ regular} \Rightarrow \operatorname{Re} \int_{\Gamma} \overline{f}^2(z) d\bar{z} = 0$$



Akibat IV.15-18

Hal.66

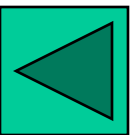
Misalkan f μ regular .

$$* \operatorname{Im} \bar{L} f = 0 \Rightarrow \int_{\Gamma} f d \bar{z} + \int_{\Gamma} \bar{f} dz = 0$$

$$* \operatorname{Im} L f = 0 \Rightarrow \int_{\Gamma} f dz + \int_{\Gamma} \bar{f} d \bar{z} = 0$$

$$* f = u + iv \Rightarrow \int_{\Gamma} f dz + \int_{\Gamma} \bar{f} d \bar{z} = 2\mu \iint_{\Omega} v dA$$

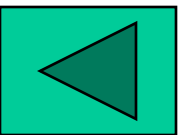
$$* \operatorname{Re} \bar{L} f = 0 \Rightarrow \int_{\Gamma} f d \bar{z} - \int_{\Gamma} \bar{f} dz = 0$$



Akibat IV.19

Hal. 67

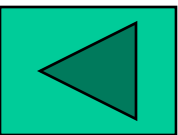
$$f \text{ } \mu \text{ regular} \Rightarrow \int_{\Gamma} f^2(z) dz + \int_{\Gamma} \overline{f}^2(z) d\bar{z} = 0$$



Akibat IV.20,21
Hal. 67

$$\left. \begin{array}{l} f = u + iv_1 \quad \mu \text{ regular} \\ g = u + iv_2 \quad \mu \text{ regular} \end{array} \right\} \Rightarrow * \operatorname{Im} \int_{\Gamma} \overline{f - g}(z) dz = 0$$

$$* \operatorname{Im} \int_{\Gamma} (f - g)(z) d\bar{z} = 0$$



Akibat IV.22,23
Hal. 67

$$\left. \begin{array}{l} f = u_1 + iv \quad \mu \text{ regular} \\ g = u_2 + iv \quad \mu \text{ regular} \end{array} \right\} \Rightarrow \begin{array}{l} * \operatorname{Re} \int_{\Gamma} \overline{(f - g)(z)} dz = 0 \\ * \operatorname{Im} \int_{\Gamma} (f - g)(z) d\bar{z} = 0 \end{array}$$



Proposisi IV.12

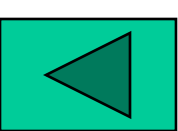
Hal. 70

Misalkan f μ regular pada cakram $|z - z_0| \leq R$, maka

$$f(z_0) = \frac{-i}{\mu\pi\Psi_1(\mu R)} \int_{\partial D(z_0, R)} \frac{\overline{f(z)}}{z - z_0} dz$$

dan

$$\frac{\partial}{\partial \bar{z}} \overline{f(z_0)} = \frac{-i}{2\mu\pi\Psi_1(\mu R)} \int_{\partial D(z_0, R)} \frac{\overline{f(z)}}{z - z_0} dz$$

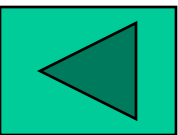


Proposisi IV.13, 14
Hal. 71

Misalkan f μ regular pada Ω , maka

$$* \int_{\partial D} g(z) dz = i \iint_{\Omega} \mu \overline{g(z)} dx dy$$

$$* g(z_0) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{g(z)}{z - z_0} dz - \frac{\mu}{2\pi} \iint_{\Omega} \frac{\overline{g(z)}}{z - z_0} dx dy, \quad z_0 \in \Omega$$



PRINSIP REFLEKSI FUNGSI μ REGULAR

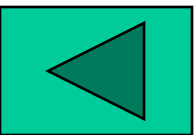
NOTASI

D adalah domain pada bidang yang simetri terhadap sumbu real.

$$D^+ = D \cap \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$$

$$D^- = \overline{\{z \in D^+\}}$$

$$I = D \cap \mathbb{R}$$



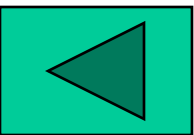
Proposisi IV.15

Hal. 72

Misalkan u fungsi kontinu pada suatu domain Ω .
Jika untuk masing - masing $x \in \Omega$ terdapat bilangan $\varepsilon > 0$ demikain sehingga untuk $0 < r < \varepsilon$ berlaku

$$u(x) = \frac{1}{2\pi I_0(\mu r)} \int_0^{2\pi} u(x + re^{i\theta}) d\theta$$

maka u panharmonik pada Ω .



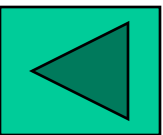
Proposisi IV.16

Hal. 73

Misalkan $u(z)$ fungsi panharmonik bernilai real pada D^+ sehingga $u(z) \rightarrow 0$ untuk $z \in D^+$ dan $z \rightarrow I$.

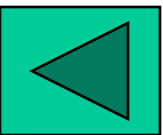
Maka $u(z)$ panharmonik pada D , dan perluasannya memenuhi

$$u(\bar{z}) = -u(z), \quad z \in D.$$



Proposisi IV.17 (Ketunggalan) Hal. 74

Misalkan f dan g dua buah fungsi μ regular pada Ω .
Maka $f \equiv g$ jika dan hanya jika himpunan infinit
 $\{z \in \Omega : f(z) \neq g(z)\}$ punya titik limit di Ω .

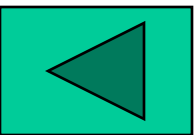


Proposisi IV. 18

Hal. 74

Misalkan f fungsi μ regular pada D^+ dan bernilai real pada sumbu real, maka f dapat diperluas atas daerah D .
Pendefinisian fungsi yang digunakan

$$F(z) = \begin{cases} f(z) & , z \in D^+ \\ \overline{f(\bar{z})} & , z \in D^- \end{cases}$$

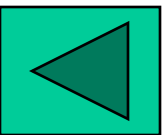


Proposisi IV.19

Hal. 74

Misalkan f fungsi μ regular pada D .

$\overline{f(z)} = f(\bar{z}), \forall z \in D \Leftrightarrow f$ bernilai real pada I .



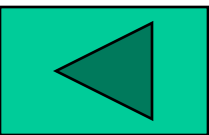
Proposisi IV.20

Hal. 75

Misalkan f fungsi μ regular pada D^+ dan bernilai imajiner pada sumbu real, maka f dapat diperluas atas daerah D .

Pendefinisian fungsi yang digunakan

$$F(z) = \begin{cases} f(z) & , z \in D^+ \\ -\overline{f(\bar{z})} & , z \in D^- \end{cases}$$

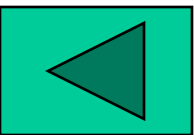


Proposisi IV.21

Hal. 75

Misalkan f fungsi μ regular pada D .

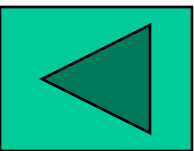
$\overline{f(z)} = -f(\bar{z}), \forall z \in D \Leftrightarrow f$ bernilai imajiner pada I .



Prinsip Maksimum Modulus Fungsi μ Regular

Proposisi IV. 22, Hal. 76

Jika f fungsi μ regular pada Ω , maka $|f|$ mencapai nilai maksimum di $\overline{\Omega}$ pada batasnya.

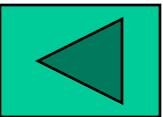


$$S = \{ f(z) = \sum_{n=0}^{\infty} a_n z^n : f \text{ analitik univalen pada } D(0,1) \\ f(0) = 0, f'(0) = 1 \}$$

Konjektur Bieberbach 1916

$$a_n \leq n$$

Bukti Louis de Branges 1984



J.L. Schiff & Walker 1990

$$\begin{aligned} F : S &\rightarrow S_\mu \\ f &\mapsto F_f, \forall f \in S \end{aligned}$$

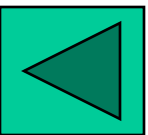
$$F_f(z) = \sum_{n=0}^{\infty} a_n z^n \Psi_n(\mu r) + \sum_{n=0}^{\infty} \overline{a_n} \frac{\mu}{2(n+1)} \overline{z^{n+1}} \Psi_{n+1}(\mu r).$$

F_f fungsi μ regular pada $D(0,1)$.

Selanjutnya kita pandang,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \Psi_n(\mu r) + \sum_{n=1}^{\infty} \overline{a_{-n}} \overline{z^n} \Psi_n(\mu r).$$

dengan $a_{-n-1} = \frac{\mu}{2(n+1)} \overline{a_n}$, $n = 0, 1, 2, 3, \dots$



Kelas fungsi Analitik Bieberbach- Eilenberg

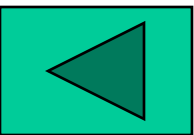
$$B = \{ f(z) = \sum_{n=1}^{\infty} a_n z^n : f(z) f(\xi) \neq 1 \text{ untuk tiap pasang titik } z \text{ dan } \xi \text{ di } D(0,1) \}$$

Roginski (1939): Konjektur $|a_n| \leq 1, \forall n$

Bukti oleh Lebedev dan Milin (1951).

Aharonov dan Nehari (1970) menunjukkan

$$\sum_{n=1}^{\infty} |a_n|^2 \leq 1.$$

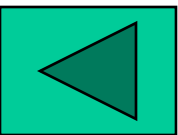


Fungsi μ Regular

Dibangun dari Kelas Fungsi Bieberbach-Eilenberg

$$* \left| a_n - a_{-n} \right| \leq 1 + \frac{\mu}{2n}, n = 1, 2, 3, \dots$$

$$* \sum_{n=1}^{\infty} \left| a_{-n} \right|^2 \leq \frac{1}{6} \left(\frac{\mu \pi}{2} \right)^2$$



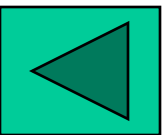
Komposisi Dua Fungsi

Hal. 79

Misalkan f dan g di $C^1(\Omega)$, dan $f \circ g$ terdefinisi pada Ω .
Maka berlaku,

$$\frac{\partial}{\partial z} (f \circ g)(z) = \frac{\partial f}{\partial z} \Big|_{g(z)} \frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}} \Big|_{g(z)} \frac{\partial \bar{g}}{\partial z}(z)$$

$$\frac{\partial}{\partial \bar{z}} (f \circ g)(z) = \frac{\partial f}{\partial z} \Big|_{g(z)} \frac{\partial g}{\partial \bar{z}}(z) + \frac{\partial f}{\partial \bar{z}} \Big|_{g(z)} \frac{\partial \bar{g}}{\partial \bar{z}}(z)$$



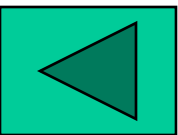
Akibat IV.30

Hal. 79

Misalkan f analitik dan g μ regular dan $f \circ g$ terdefinisi pada Ω .
Maka berlaku,

$$\frac{\partial}{\partial z} (f \circ g)(z) = \frac{\partial f}{\partial z} \bigg|_{g(z)} \frac{\partial g}{\partial z}(z)$$

$$\frac{\partial}{\partial \bar{z}} (f \circ g)(z) = \frac{\partial f}{\partial \bar{z}} \bigg|_{g(z)} \frac{\partial g}{\partial \bar{z}}(z)$$

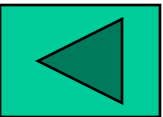


Proposisi IV.23

Hal. 80

Misalkan f fungsi μ regular disetiap titik pada suatu daerah yang memiliki batas sejumlah hingga kontur - kontur tutup sederhana Γ . Misalkan pula $\Phi(w)$ fungsi analitik di $w = f(z)$, maka

$$\operatorname{Re} \int_{\Gamma} \Phi(f) f_x - \mu \operatorname{Im}(\Phi(f) \bar{f}) dz = 0.$$

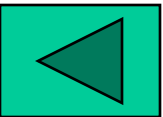


Proposisi IV.24

Hal. 80

Misalkan f fungsi μ regular di dalam dan pada kontur tutup tutup sederhana Γ dan $f(z) \neq 0$ untuk setiap titik pada Γ . Maka jumlah total bilangan nol dari $f(z)$ di dalam Γ adalah

$$N = \operatorname{Re} \left(\frac{1}{2\pi i} \int_{\Gamma} \left(\frac{f_x}{f} + \frac{\mu}{2} \frac{f}{\overline{f}} - \frac{\mu}{2} \frac{\overline{f}}{f} \right) dz \right).$$



TERIMAKASIH