Panharmonic Functions

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Summary

In this paper we discuss some properties of panharmonic functions those are similar to harmonic functions. In particular there are generalisations of harmonic functions, Liouville's theorem, and convergence in the mean theorem. By using Green's Identity, the uniqueness theorem is shown to produce another generalised harmonic function.

Abstrak

Tulisan ini membahas beberapa sifat fungsi panharmonik sebagai hasil generalisasi dari sifat fungsi harmonik. Beberapa sifat fungsi panharmonik diperoleh dengan memanfaatkan sifat korespondensi satu-satu antara kelas fungsi panharmonik di bidang dengan sub kelas fungsi harmonik di ruang. Selain itu Ketunggalan fungsi panharmonik dijelaskan dengan memanfaatkan identitas Green.

1. Introduction

In this paper we establish some results relating to the solutions of the Yukawa equation, i.e.:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mu^2 u, \qquad (1)$$

μ is a positive constant. A C^2-solution of (1) in a domain $\Omega \subseteq \Re^2$ is called *panharmonic*. Equation (1) arose out of an attempt by the Japanese physicist Hideki Yukawa to describe nuclear potential of a point charge as $e^{-\mu r}/r$. The resulting potential distribution satisfies the 3-dimensional version of equation (1). A comprehensive

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Yukawan potential theory has subsequently developed by Duffin [1], J. L. Schiff & W.J. Walker [2], to which we refer to the reader.

This paper will inform many other properties of real panharmonic functions. Many of properties will be proved by using correspondence properties between panharmonics functions in two space with a subclass of harmonic functions in three space. Further, we will show that solution of equation (1) depends on the boundary value. It is said that some of properties in harmonic function is also satisfied by panharmonic functions.

In this paper, the function u is said to be panharmonic in a closed domain Ω, if it is panharmonic in the interior and continuous on the boundary of $\Omega(=\partial\Omega)$. Moreover, we assume that Ω is a compact region in $\Re^2$ and $|\nabla u|^2$ is integrable on Ω.

2. **Properties of Panharmonic Functions**

Panharmonic functions in two spaces are in one to one correspondence with a subclass of harmonic functions in three spaces. Thus given that *u(x,y)* is *panharmonic*, this correspondence is defined by the mapping

$$v(x,y,z) = \cos\mu z \, u(x,y). \qquad (2)$$

So,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0, \qquad (3)$$

therefore v is *a harmonic function*.

Conversely , if *v(x,y)* is a harmonic function, then

$$U(x,y,z)=v(x,y)\cosh\mu z \qquad (4)$$

is a panharmonic function, because

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=\mu^2 U. \qquad (5)$$

The virtue of this correspondence is that *panharmonic functions* may inherit many of the well-known properties of *harmonic functions*.

Theorem 1

If $u(x,y)$ is panharmonic non positive in Ω and if for a constant $M < 0$,

$$u(x,y) \geq M \text{ at boundary points of } \Omega, \tag{6}$$

then $u(x,y) > M$ at interior points of Ω. (7)

Proof

Consider the cylindrical region H of three space, defined as $(x,y) \subseteq \Omega$ and

$$\frac{-\pi}{2\mu} \leq z \leq \frac{\pi}{2\mu}. \tag{8}$$

Let's first define $V(x,y,z) = u(x,y)\cos \mu z$ on H. It is clear that V is a harmonic function, on the top and on the bottom of this cylinder the corresponding harmonic function V vanishes. On the sides of cylinder $-V(x,y,z) \leq -M$. Because V is a harmonic function, it also $-V$. Thus by the maximum principle for harmonic functions, it follows that at interior points of the cylinder H, $-V(x,y,z) < -M$. (9)

Unless, the possibly of $-V$ is constant. But if V is constant, then $V=0$, because V vanishes on the top and bottom of this cylinder. Thus (9) is always true. Then $V(x,y,0) = u(x,y)$, so if (x,y) is an interior point of Ω, then we see that (7) follows from (9), and the proof is complete.

Example

Let $u(x,y)=\cosh(\mu(x+y)/\sqrt{2})$ in the circular disk $x^2 + y^2 \leq a^2$. It is clear that u is a positive panharmonic function on the disk. We can see that the maximum of u occurs on the boundary, but the minimum of u occurs on the straight line $x+y=0$. Thus, in general, the maximum principle for harmonic functions is not hold in the panharmonic functions.

Theorem 2

Let $u_1, u_2, u_3, \ldots$ be an infinite sequences of panharmonic functions in Ω, and convergent in the mean square in Ω. Then the sequence converges uniformly in any closed region Ω' interior to Ω, to a panharmonic limiting function.

Proof

Let u_1, u_2, u_3, ... be an infinite sequences of panharmonic functions in Ω, and convergent in the mean square in Ω to *u*. Consider the cylindrical region *H* of three space, defined as $(x,y)\subseteq\Omega$ and $\frac{-\pi}{\mu}\le z\le\frac{\pi}{\mu}$, and the cylindrical region *H'* of three space, defined as $(x,y)\subseteq\Omega'\subset\Omega$ and $\frac{-\pi}{2\mu}\le z\le\frac{\pi}{2\mu}$. Let us define

$V_n(x,y,z)=u_n(x,y)\cos\mu z$ on cylinder *H*.

V_n is convergent in the mean square in *H* to $u(x,y)\cos\mu z$, because

$$\int_H \left|u_n(x,y)\cos\mu z-u(x,y)\cos\mu z\right|^2 dV\le\int_H\left|u_n(x,y)-u(x,y)\right|^2\left|\cos\mu z\right|^2 dV$$
$$\le\int_H\left|u_n(x,y)-u(x,y)\right|^2 dV.$$

By applying theorem ([2],p.268), we obtain the sequence is uniformly convergent in *H*'. Then, putting $z = 0$, gives u_n is uniformly convergent in Ω'.

Theorem 3

If (u_m) is a sequence of panharmonic functions on an open set Ω that is uniformly bounded on each compact subset of Ω, then some subsequence of (u_m) converges uniformly on each compact subset of Ω.

Proof

Suppose $\Omega'\subset\Omega$ is compact. Construct the cylindrical domain *H* of three space, defined as $(x,y)\subseteq\Omega$ and $\frac{-\pi}{\mu}<z<\frac{\pi}{\mu}$, and the cylindrical region *H'* of three space, defined as $(x,y)\subseteq\Omega'$ and $\frac{-\pi}{2\mu}\le z\le\frac{\pi}{2\mu}$. Let us define $V_m(x,y,z)=u_m(x,y)\cos\mu z$ on cylinder *H*.

Then $\left|V_m(x,y,z)\right|=\left|u_m(x,y)\cos\mu z\right|\le\left|u_m(x,y)\right|\le M$, for some positives real number *M*. Clearly V_m is a sequence of harmonic functions on *H* that is uniformly bounded on each compact subset of *H*. By the theorem ([4], p.35), the some subsequence of (V_m) converges uniformly on compact subset *H'* in *H* . Say that (V_{mn}) converges uniformly to *V*. Putting $z=0$, we obtain $V_{mn}(x,y,0)=u_{mn}(x,y)$ uniform converges to $V(x,y,0)$ on Ω'.

Theorem 4

A bounded panharmonic function on $\Re^2$ vanishes.

Proof

Suppose $u(x,y)$ is a panharmonic function on $\Re^2$, bounded by M.

Define $V(x,y,z) = u(x,y)\cos\mu z$ on $\Re^3$. Clearly V is a harmonic function on $\Re^3$ and bounded by M. By the Liouville theorem's (see [4], p.31), V is a constant function. But for $z= \pi/2\mu$ and (x,y) any points on $\Re^2$, $V = 0$. Therefore, V must vanish for any point on $\Re^3$. Thus, for any (x,y) on $\Re^2$ and $z = 0$, then $V(x,y,0)=u(x,y)=0$. We conclude that u is a vanishes function on $\Re^2$.

The next result shows that a continuous function on upper half-space that is bounded and panharmonic on an open half space is determined by its boundary values.

Corollary 1

Let $H_2 = \{(x,y) \in \Re^2 : y>0\}$. Supposed $u(x,y)$ is a continuous bounded function on $\overline{H_2}$ that is panharmonic on H_2. If $u=0$ on ∂H_2, then $u \equiv 0$ on $\overline{H_2}$.

Proof

Define $V(x,y,z) = u(x,y)\cos\mu z, z\in R$. Clearly V is a harmonic function on subset of $\Re^3$, where x, z in $\Re$ and $y > 0$, and $V(x,0,z) = 0$. By the corollary ([4], p. 32), putting $z = 0$, then we obtain $V(x,y,0)=u(x,y)=0$, for any points (x,y) on $\Re^2$, where $y > 0$. Thus, $u(x,y)= 0$, for any points on $\overline{H_2}$.

Next, the following theorems are consequences of the first and second Green identity. Let Ω be a bounded domain with boundary $\partial\Omega$ and let n denote the unit outward normal to $\partial\Omega$. Let u and v be $C^2(\Omega)$ functions. We obtain Green's first Identity :

$$\int_\Omega v\Delta u dV + \int_\Omega \nabla u.\nabla v dV = \int_{\partial\Omega} v\frac{\partial u}{\partial n} dS.$$

Interchanging u and v above and subtracting the first equation from the second, we obtain Green's Second Identity :

$$\int_\Omega (v\Delta u - u\Delta v) dV = \int_{\partial\Omega} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) dS.$$

Theorem 5

If U is panharmonic and vanishes at all points of the boundary of Ω, it vanishes at all points of Ω.

Proof

Consider the Green's First Identity, and let's identify V with U, and suppose U is panharmonic. The Green's identity then becomes

$$\int_{\Omega} \left(u^2 U^2 + |\nabla U|^2 \right) dV = \int_{\partial\Omega} U \frac{\partial U}{\partial n} dS = 0 \qquad (10)$$

Since the right side is vanish, U^2 and $|\nabla U|^2$ are integrable in Ω, and never negatives, it must be vanished at all point of Ω.

We deduce at once an important consequence. Let us suppose that u and v are both panharmonic in Ω, and take on the same boundary values. Then their difference is panharmonic in Ω and reduces to 0 on the boundary. Hence it vanishes throughout Ω. We could state the result as follows.

Theorem 6

A panharmonic function in Ω is uniquely determined by its values on the boundary of Ω.

The surface integral in equation (10) will also vanish if the normal derivative vanishes everywhere on ∂Ω. Again we see that as a consequence, u will vanish in Ω.

Theorem 7

If U is one value, function, panharmonic in Ω, and if its normal derivative vanishes at every points of the boundary of Ω, then U vanishes in Ω.

Theorem 8

Let U be panharmonic in Ω, and satisfies the condition on the boundary

$$\frac{\partial U}{\partial n} + hU = g, \qquad (11)$$

where h and g are continuous functions on $\partial\Omega$, and h is never negative. Then there is no different function satisfying the same conditions.

Proof

Let us suppose that U and V are both panharmonic, and satisfy the same condition on the boundary as equation (11).Then their difference is a panharmonic in Ω and

$\frac{\partial(U-V)}{\partial n}+h(U-V)=0.$ The Green identity becomes

$$\int_{\Omega}\left(\mu^2(U-V)^2+\left|\nabla(U-V)\right|^2\right)dV+\int_{\partial\Omega}h(U-V)^2dS=0.$$

Since $(U-V)^2$ and $|\nabla(U-V)|^2$ are integrable in Ω, and h is a continuous function on $\partial\Omega$, and never negative, it must vanish at all point of Ω. Therefore, $U=V$ in Ω.

Theorem 9

If U and V panharmonic in Ω, then $\int_{\partial\Omega}(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n})dS=0.$

Proof

Let U and V are panharmonic in Ω . Apply the second Green identity, we have

$$\int_{\Omega}(V\mu^2U+\nabla U.\nabla V)dV=\int_{\partial\Omega}V\frac{\partial U}{\partial n}dS,\ \text{and}$$

$$\int_{\Omega}(U\mu^2V+\nabla V.\nabla U)dV=\int_{\partial\Omega}U\frac{\partial V}{\partial n}dS.$$

If we subtract both two equations above, then we obtain the result.

References

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HASIL PEMANTAUAN TUTORIAL ANALISIS

No	N a m a	Kog	Bel	Kom	Sikap	Hadir

1	Siti Rahmah Nurshiami	5.2	5.2	4	5	8
2	Siti Sakirah Arsyad	5	4.6	3.8	3.4	8
3	Iwan Budiman	4.75	4.25	3.5	3.35	4
4	Budi Rudianto	6	6	6	6	4

P = 3, F = 4, G = 6, E = 7

Catatan

1. Komunikasi dan inisiatif masih kurang. (cukup rajin)
2. Tulisan, komunikasi, inisiatif masih kurang. (cukup rajin)
3. Tulisan, komunikasi, inisiatif masih kurang. (kurang rajin)
4. Komunikasi lumayan, inisiatif masih kurang.

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Pengamat

Endang Cahya